

EMBEDDINGS OF VECTOR-VALUED BERGMAN SPACES

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ABSTRACT. We show that a dyadic version of the Carleson embedding theorem for the Bergman space extends to vector-valued functions and operator-valued measures. This is in contrast to a result by Nazarov, Treil, Volberg in the context of the Hardy space. We also discuss some embeddings for analytic vector-valued functions.

1. INTRODUCTION

Let \mathbb{D} , \mathbb{T} denote the unit disc, respectively the unit circle, in the complex plane. We denote by \mathcal{D} the family of dyadic arcs $I \subseteq \mathbb{T}$, i.e. sets of the form

$$I = \{e^{it} : \frac{2\pi k}{2^n} \leq t \leq \frac{2\pi(k+1)}{2^n}\}, \quad k = 0, 1, \dots, 2^n - 1,$$

for $n \in \mathbb{N}$. For each arc I , we denote its length by $l(I)$ and we consider the corresponding Carleson "square" $Q_I = \{z \in \mathbb{D} : z/|z| \in I, 1 - l(I)/2\pi \leq |z|\}$. Let T_I denote the upper half of Q_I , i.e. $T_I = \{z \in Q_I : |z| < 1 - l(I)/4\pi\}$.

Given a positive Borel measure μ on \mathbb{D} , a classical theorem by Carleson states that the Hardy space H^2 is continuously contained in $L^2(d\mu)$ if and only if there exists a constant $c > 0$ such that $\mu(Q_I) \leq cl(I)$, for all $I \in \mathcal{D}$, or, equivalently, the embedding operator

$$Df(z) := \sum_{I \in \mathcal{D}} \left(\frac{1}{l(I)} \int_I f dm \right) \chi_{T_I}(z), \quad z \in \mathbb{D},$$

is bounded from $L^2(dm)$ to $L^2(d\mu)$, where m denotes the Lebesgue measure on \mathbb{T} (see [5]). This last equivalent formulation of the result is also called the dyadic Carleson embedding theorem.

A result by Nazarov, Treil, Volberg [5] shows that the dyadic Carleson embedding theorem does not extend to Hilbert space-valued functions and operator-measures, while sharp dimension-dependent estimates for matrix-valued measures are provided in [6]. More precisely, given an n -dimensional Hilbert space H and a positive $n \times n$ matrix-valued Borel measure μ on \mathbb{D} , the operator D (defined analogously for vector-valued functions) is bounded from $L^2(H, dm)$ to $L^2(H, d\mu)$ if and only if μ has finite Carleson intensity $\|\mu\|_C$ (see [6] for the definition) and the following *sharp* dimensional estimate holds

$$(1.1) \quad \|D\| \lesssim C \log n \|\mu\|_C^{1/2}.$$

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Here $L^2(H, dm)$ denotes the space of H -valued functions that are square integrable with respect to m on \mathbb{T} , while $L^2(H, d\mu)$ represents the space of H -valued functions that are square integrable with respect to μ on \mathbb{D} .

This result is one in a series of deep investigations (see [7, 6, 4]), which reveal one and the same phenomenon: natural generalizations of some basic results from the scalar-valued setting to vector-valued Hardy spaces continue to hold as long as the "target-space" has finite dimension, with the involved estimates depending on this dimension in such a way that the results fail in the infinite-dimensional case. Relevant examples that illustrate this behaviour appear in the study of the Riesz projection, respectively of Hankel operators with operator-valued symbols on vector-valued Hardy spaces. Remarkably enough, when considering the analogous problems for the Bergman projection, respectively for Hankel operators on vector-valued Bergman spaces, one encounters a completely different situation, namely, in this context the scalar results have natural generalizations to the vector-valued setting that are independent of the dimension of the "target space" (see [1, 2]).

In this note we show that, when considering Carleson embeddings for vector-valued Bergman spaces, one finds a behaviour consistent with the one described above. We start with the observation that the disc-analogue of the dyadic Carleson embedding theorem continues to hold for operator-valued measures. Subsequently, we use techniques developed in [1] in order to discuss some embeddings for analytic vector-valued functions. More precisely, we characterize the non-negative operator-valued functions $G : \mathbb{D} \rightarrow \mathcal{B}(H)$ for which the vector-valued Bergman spaces with operator-valued Békollé-Bonami weights are continuously contained in $L^2(H, G dA)$, where A denotes the normalized Lebesgue area measure on \mathbb{D} . The obtained characterizations are independent of dimension, which is mainly due to the fact that the Carleson condition expressed in terms of Carleson "squares" is equivalent to the analogue condition for top halves of Carleson "squares". We would like to emphasize that our Carleson embedding theorem for analytic functions is equivalent to its dyadic version even when the target space H has infinite dimension. We refer to [3] for related results concerning multipliers on vector valued Bergman spaces. Finally, we present a short application of our embedding theorem to integration operators of Volterra type with operator-valued symbols.

2. A REMARK ON DYADIC CARLESON EMBEDDINGS

Let us first introduce some notation. For two real-valued (respectively, positive operator-valued) functions E_1, E_2 we write $E_1 \sim E_2$, or $E_1 \lesssim E_2$, if there exists a positive constant k independent of the argument such that $\frac{1}{k}E_1 \leq E_2 \leq kE_1$, respectively $E_1 \leq kE_2$.

We denote by A the normalized Lebesgue area measure on \mathbb{D} . Let H be a Hilbert space and consider a positive operator-valued Borel measure μ on \mathbb{D} , i.e. a countably additive function defined on the Borel measurable subsets of \mathbb{D} with values in the set of non-negative operators on H . We define its Carleson

intensity as follows

$$\|\mu\|_C := \sup \frac{\|\mu(Q_I)\|}{A(Q_I)} = \sup_{I \in \mathcal{D}} \sup_{e \in H, \|e\|=1} \frac{\langle \mu(Q_I)e, e \rangle}{A(Q_I)}$$

We consider the following analogue for the disc of the embedding operator D defined above

$$Bf(z) := \sum_{I \in \mathcal{D}} \left(\frac{1}{A(T_I)} \int_{T_I} f dA \right) \chi_{T_I}(z), \quad z \in \mathbb{D}, \quad f \in L^2(H, dA).$$

We are concerned with the boundedness of B from $L^2(H, dm)$ to $L^2(H, d\mu)$. As pointed out in [5], defining the integral $\int_{\mathbb{D}} \langle d\mu(z)f(z), f(z) \rangle$ for an operator-valued measure μ is, in general, a delicate issue. However, since Bf is a vector-valued "step-function", one has

$$\int_{\mathbb{D}} \langle d\mu(z)(Bf)(z), (Bf)(z) \rangle = \sum_{I \in \mathcal{D}} \langle \mu(T_I) \left(\frac{1}{A(T_I)} \int_{T_I} f dA \right), \left(\frac{1}{A(T_I)} \int_{T_I} f dA \right) \rangle.$$

The classical Carleson embedding theorem for the scalar valued Bergman space L_a^2 states that, given a positive scalar Borel measure μ on \mathbb{D} , the space L_a^2 is continuously contained in $L^2(d\mu)$ if and only if $\|\mu\|_C < \infty$, or, equivalently the operator B is bounded from $L^2(dA)$ to $L^2(d\mu)$.

In contrast to the dimension-depending sharp estimate (1.1) proven in [6], it turns out that, in our setting, we have

Remark 2.1. *Assume $\dim H \leq \infty$. The operator B is bounded from $L^2(H, dA)$ to $L^2(H, d\mu)$ if and only if μ has finite Carleson intensity. In addition, there are absolute constants $k_1, k_2 > 0$ such that the following inequalities hold*

$$k_1 \|\mu\|_C^{1/2} \leq \|B\| \leq k_2 \|\mu\|_C^{1/2}.$$

Proof. The proof is rather straight-forward, but we include it for the sake of completeness. Assume first that $\|\mu\|_C < \infty$. Notice that $\mu(T_I) \leq \mu(Q_I)$ and that $A(T_I) \sim A(Q_I)$, for $I \in \mathcal{D}$. Then, for $f \in L^2(H, dA)$, we have

$$\begin{aligned} \int_{\mathbb{D}} \langle d\mu(z)(Bf)(z), (Bf)(z) \rangle &= \sum_{I \in \mathcal{D}} \langle \mu(T_I) \left(\frac{1}{A(T_I)} \int_{T_I} f dA \right), \left(\frac{1}{A(T_I)} \int_{T_I} f dA \right) \rangle \\ &\lesssim \|\mu\|_C \sum_{I \in \mathcal{D}} \frac{1}{A(T_I)} \left\| \int_{T_I} f dA \right\|^2 \\ &\leq \|\mu\|_C \sum_{I \in \mathcal{D}} \int_{T_I} \|f\|^2 dA \\ &= \|\mu\|_C \int_{\mathbb{D}} \|f\|^2 dA, \end{aligned}$$

by the Cauchy-Schwarz inequality.

Conversely, suppose B is bounded. For every $e \in H$ and $I \in \mathcal{D}$ we have $B(\chi_{T_I}e) = \chi_{T_I}e$, and hence

$$\langle \mu(T_I)e, e \rangle = \|B(\chi_{T_I}e)\|_{L^2(H, d\mu)}^2 \leq \|B\|^2 \|\chi_{T_I}e\|_{L^2(H, dA)}^2 = \|B\|^2 A(T_I) \|e\|^2,$$

which implies

$$\alpha := \sup_{I \in \mathcal{D}} \frac{\|\mu(T_I)\|}{A(T_I)} \leq \|B\|^2.$$

As in the scalar case, it is easily seen that α is comparable with $\|\mu\|_C$. Indeed, for any $I \in \mathcal{D}$ we can write the Carleson "square" Q_I as an infinite union of top halves of Carleson "squares" corresponding to the dyadic subintervals of I ,

$$Q_I = \cup_{n=0}^{\infty} \cup_{k=1}^{2^n} T_{l(I)/2^n}^k,$$

with $A(T_{l(I)/2^n}^k) \sim (\frac{l(I)}{2^n})^2$ for $n \in \mathbb{N}$ and $1 \leq k \leq 2^n$. This implies

$$\begin{aligned} \|\mu(Q_I)\| &\leq \sum_{n=0}^{\infty} \sum_{k=1}^{2^n} \|\mu(T_{l(I)/2^n}^k)\| \leq \alpha \sum_{n=0}^{\infty} \sum_{k=1}^{2^n} A(T_{l(I)/2^n}^k) \\ &\lesssim \alpha \sum_{n=0}^{\infty} \sum_{k=1}^{2^n} (\frac{l(I)}{2^n})^2 \lesssim \alpha l(I)^2 \lesssim \alpha A(Q_I). \end{aligned}$$

The inequality $\|\mu\|_C \gtrsim \alpha$ is trivial. \square

3. EMBEDDINGS OF VECTOR-VALUED BERGMAN SPACES

We begin by introducing the concepts we are working with.

Given a separable Hilbert space H , we denote by $\mathcal{B}(H)$ the space of bounded linear operators on H . We say that a nonnegative operator-valued function $G : \mathbb{D} \rightarrow \mathcal{B}(H)$ (i.e. $G(z)$ is a nonnegative operator on H , a.e. $z \in \mathbb{D}$) is *integrable* on \mathbb{D} if for any $x, y \in \mathcal{H}$ the scalar function

$$z \mapsto \langle G(z)x, y \rangle$$

is integrable on \mathbb{D} and we have

$$\left| \int_{\mathbb{D}} \langle G(z)x, y \rangle dA(z) \right| \leq C \|x\| \|y\|,$$

for some constant C independent of x and y . The bounded linear operator defined this way will be denoted by $\int_{\mathbb{D}} G dA$.

For $\eta > -1$, we denote $dA_{\eta}(z) = (\eta+1)(1-|z|)^{\eta} dA(z)$. We consider operator-valued weights $W : \mathbb{D} \rightarrow \mathcal{B}(\mathcal{H})$ such that

- (1) $W(z)$ is a nonnegative operator that is invertible a.e. $z \in \mathbb{D}$;
- (2) $(1-|z|^2)^{\eta} W$ and $(1-|z|^2)^{\eta} W^{-1}$ are integrable on \mathbb{D} ;
- (3) $\int_{\mathbb{D}} W dA_{\eta}$ is invertible.

For such a weight W the corresponding L^2 space on \mathbb{D} is denoted $L^2(H, W dA_{\eta})$ and is endowed with the norm

$$\|f\|^2 = \int_{\mathbb{D}} \langle W(z)f(z), f(z) \rangle dA_{\eta}(z).$$

We say that the weight W belongs to the so-called Békollé-Bonami class $B_2(\eta)$ ($\eta > -1$) if

$$\sup_S \left\| \left(\frac{1}{A_{\eta}(S)} \int_S W dA_{\eta} \right)^{1/2} \left(\frac{1}{A_{\eta}(S)} \int_S W^{-1} dA_{\eta} \right)^{1/2} \right\| < \infty,$$

where the supremum is taken over all Carleson squares:

$S = \{z = re^{it} : 1 - h < r < 1, |t - \theta| < \pi h\}$, with $h \in (0, 1)$, $\theta \in [0, 2\pi)$.

For $W \in B_2(\eta)$ the subspace of $L^2(H, WdA_\eta)$ consisting of H -valued analytic functions in \mathbb{D} is closed in $L^2(H, WdA_\eta)$ (see [1]). We denote this subspace by $L_a^2(H, WdA_\eta)$.

We shall also consider the "average" version of W on the discs

$$D_{z,r} = \{\zeta \in \mathbb{D} : |z - \zeta| < r(1 - |z|)\}, \quad r \in (0, 1),$$

given by

$$[W]_{z,r} := \frac{1}{A(D_{z,r})} \int_{D_{z,r}} W dA, \quad z \in \mathbb{D}.$$

If W belongs to $B_2(\eta)$ ($\eta > -1$), it follows that $[W]_{z,r}$ is invertible, for all $z \in \mathbb{D}$ (see Proposition 3.2 in [1]).

We are now ready to characterize the nonnegative operator-valued functions G for which $L_a^2(H, WdA_\eta)$ is continuously embedded in $L^2(H, GdA)$. Our result is actually slightly more general, in the sense that we consider embeddings for derivatives of functions in $L_a^2(H, WdA_\eta)$.

Theorem 3.1. *Assume $G : \mathbb{D} \rightarrow \mathcal{B}(H)$ is an integrable nonnegative operator-valued function, let W be an operator-valued weight that belongs to $B_2(\eta)$ for some $\eta > -1$ and let $r \in (0, 1)$. Then, for any nonnegative integer n ,*

$$(3.2) \quad \int_{\mathbb{D}} \langle G(z)f^{(n)}(z), f^{(n)}(z) \rangle dA(z) \lesssim \int_{\mathbb{D}} \langle W(z)f(z), f(z) \rangle dA_\eta(z), \quad f \in L_a^2(H, WdA_\eta),$$

holds if and only if

$$(3.3) \quad \int_{D_{\lambda,r}} G dA \lesssim (1 - |\lambda|)^{2n} \int_{D_{\lambda,r}} W dA_\eta,$$

where the constant involved in the last inequality above is independent of λ .

Proof. Assume (3.3) holds. For each $z \in \mathbb{D}$, we apply Cauchy's formula to the analytic function $\zeta \mapsto G^{1/2}(z)f(\zeta)$ to obtain the estimate

$$\langle G(z)f^{(n)}(z), f^{(n)}(z) \rangle \lesssim \frac{1}{(1 - |z|)^{2(n+1)}} \int_{D_{z,\beta}} \langle G(z)f(\zeta), f(\zeta) \rangle dA(\zeta),$$

where $\beta \in (0, 1)$. Now integrate the above inequality on \mathbb{D} to deduce

$$\int_{\mathbb{D}} \langle G(z)f^{(n)}(z), f^{(n)}(z) \rangle dA(z) \lesssim \int_{\mathbb{D}} (1 - |z|)^{-2(n+1)} \int_{D_{z,\beta}} \langle G(z)f(\zeta), f(\zeta) \rangle dA(\zeta) dA(z).$$

Notice that, for $0 < \beta < \frac{r}{r+1}$, the inclusion $D_{z,\beta} \subseteq \tilde{D}_{z,r} := \{\zeta \in \mathbb{D} : |\zeta - z| < r(1 - |\zeta|)\}$ holds. Using this together with Fubini's theorem in the last relation

above we obtain

$$\begin{aligned}
\int_{\mathbb{D}} \langle G(z) f^{(n)}(z), f^{(n)}(z) \rangle dA(z) &\lesssim \int_{\mathbb{D}} (1 - |z|)^{-2(n+1)} \int_{\tilde{D}_{z,r}} \langle G(z) f(\zeta), f(\zeta) \rangle dA(\zeta) dA(z) \\
&\lesssim \int_{\mathbb{D}} \langle \int_{D_{\zeta,r}} G(z) dA(z) f(\zeta), f(\zeta) \rangle (1 - |\zeta|)^{-2(n+1)} dA(\zeta) \\
&\lesssim \int_{\mathbb{D}} \langle \int_{D_{\zeta,r}} W dA_{\eta} f(\zeta), f(\zeta) \rangle (1 - |\zeta|)^{-2} dA(\zeta) \\
&\lesssim \int_{\mathbb{D}} \langle [W]_{\zeta,r} f(\zeta), f(\zeta) \rangle dA_{\eta}(\zeta),
\end{aligned}$$

by (3.3) and since $(1 - |z|) \sim (1 - |\zeta|)$ for $z \in D_{\zeta,r}$. Finally, an application of the second part of Theorem 3.1 in [1] now yields

$$\int_{\mathbb{D}} \langle G(z) f^{(n)}(z), f^{(n)}(z) \rangle dA(z) \lesssim \int_{\mathbb{D}} \langle W(\zeta) f(\zeta), f(\zeta) \rangle dA_{\eta}(\zeta).$$

Conversely, assume that (3.2) holds. For $e \in H$ and $\gamma > \eta$, we put $f(z) = K_{\lambda}^{\gamma}(z)e = \frac{1}{(1-\lambda z)^{\gamma+2}}e$ in (3.2) and use Proposition 3.1 in [1] to get

$$(3.4) \quad \int_{\mathbb{D}} \langle G(z)e, e \rangle |\partial_z^n K_{\lambda}^{\gamma}(z)|^2 dA(z) \lesssim \int_{\mathbb{D}} \langle W(z)e, e \rangle |K_{\lambda}^{\gamma}(z)|^2 dA_{\eta}(z) \lesssim (1 - |\lambda|)^{-2\gamma+\eta-2} \langle [W]_{\lambda,r} e, e \rangle.$$

Let $\delta := \frac{r}{r+2}$. For $|\lambda| \geq \delta > 0$ we have

$$\begin{aligned}
\int_{\mathbb{D}} \langle G(z)e, e \rangle |\partial_z^n K_{\lambda}^{\gamma}(z)|^2 dA(z) &\gtrsim \int_{D_{\lambda,r}} \langle G(z)e, e \rangle \frac{1}{|1 - \bar{\lambda}z|^{2(\gamma+n+2)}} dA(z) \\
&\gtrsim (1 - |\lambda|)^{-2\gamma-2n-4} \int_{D_{\lambda,r}} \langle G(z)e, e \rangle dA(z).
\end{aligned}$$

Combining this with (3.4), we obtain (3.3) for $|\lambda| \geq \delta$. It is easy to see that (3.3) holds for $|\lambda| < \delta$. Indeed, in order to prove (3.3) for $|z| < \delta$, it is enough to show that

$$\int_{D_{z,r}} G dA \lesssim \int_{D_{z,r}} W dA, \quad |z| < \delta,$$

or, equivalently,

$$\|(\int_{D_{z,r}} G dA)^{1/2}(\int_{D_{z,r}} W dA)^{-1/2}\| \lesssim 1, \quad |z| < \delta,$$

where the invertibility of $\int_{D_{z,r}} W dA$ follows by Proposition 3.1 in [1]. Now

$$\|(\int_{D_{z,r}} G dA)^{1/2}\| \leq \|(\int_{\mathbb{D}} G dA)^{1/2}\|.$$

Notice that, since $r(1 - |z|) > r(1 - \delta) = 2\delta$, we have $D(0, \delta) \subset D_{z,r}$. Hence

$$\|(\int_{D_{z,r}} W dA)^{-1/2}\| \leq \|(\int_{D(0,\delta)} W dA)^{-1/2}\|$$

and the desired inequality is proven. \square

Remark 3.1. Take $W = I$, $n = 0$ in Theorem 3.1 and let $\mu = G dA$ in Remark 2.1. Notice that, for these choices, condition (3.3) is equivalent to $\|\mu\|_C < \infty$, which shows that in this case the Carleson embedding from Theorem 3.1 is equivalent to its dyadic version provided in Remark 2.1. Indeed, the equivalence of the two conditions follows, as in the scalar case, by a purely geometric argument: for each fixed $r \in (0, 1)$, we can cover any disc $D_{\lambda,r}$ by a finite number (depending only on r) of top halves T_I , with $I \in \mathcal{D}$, of comparable diameter, and vice-versa.

We can now apply Theorem 3.1 to characterize the boundedness of integration operators of Volterra type acting on vector-valued Bergman spaces with Békollé-Bonami weights. For an analytic operator-valued function $G : \mathbb{D} \rightarrow \mathcal{B}(H)$, and for analytic functions $f : \mathbb{D} \rightarrow H$, we define the operator

$$T_G f(z) = \int_0^z G'(\zeta) f(\zeta) d\zeta, \quad z \in \mathbb{D}.$$

Corollary 3.1. Assume $W : \mathbb{D} \rightarrow \mathcal{B}(H)$ is an operator-valued weight that belongs to $B_2(\eta)$ for some $\eta > -1$ and let $r \in (0, 1)$. Then T_G is bounded on $L_a^2(H, W dA_\eta)$ if and only if

$$(3.5) \quad \sup_{\lambda \in \mathbb{D}} (1 - |\lambda|) \| [W]_{\lambda,r}^{1/2} G'(\lambda) [W]_{\lambda,r}^{-1/2} \| < \infty.$$

Proof. For $f \in L_a^2(H, W dA_\eta)$ we apply Theorem 3.2 in [1] to deduce

$$\begin{aligned} \|T_G f\|^2 &\sim \int_{\mathbb{D}} \langle W(z) G'(z) f(z), G'(z) f(z) \rangle (1 - |z|^2)^2 dA_\eta(z) \\ &= \int_{\mathbb{D}} \langle G'^*(z) W(z) G'(z) f(z), f(z) \rangle (1 - |z|^2)^2 dA_\eta(z). \end{aligned}$$

By Theorem 3.1 it follows that T_G is bounded on $L_a^2(H, W dA_\eta)$ if and only if

$$\int_{D_{\lambda,r}} G'^*(z) W(z) G'(z) (1 - |z|^2)^2 dA_\eta(z) \lesssim \int_{D_{\lambda,r}} W(z) dA_\eta(z).$$

As shown in Theorem 3.1 in [1], the weight $z \mapsto [W]_{z,r}$ provides an equivalent norm for $L_a^2(H, W dA_\eta)$, and hence we can replace W by $[W]_{z,r}$ above to obtain that T_G is bounded on $L_a^2(H, W dA_\eta)$ if and only if

$$(3.6) \quad \int_{D_{\lambda,r}} G'^*(z) [W]_{z,r} G'(z) (1 - |z|^2)^2 dA_\eta(z) \lesssim \int_{D_{\lambda,r}} [W]_{z,r} dA_\eta(z), \quad \lambda \in \mathbb{D}.$$

According to Remark 3.2 in [1], we have

$$(3.7) \quad k_1 [W]_{\lambda,r} \leq [W]_{z,r} \leq k_2 [W]_{\lambda,r}, \quad z \in D_{\lambda,r},$$

where the constants $k_1, k_2 > 0$ are independent of $z, \lambda \in \mathbb{D}$. From this we deduce that (3.6) is in its turn equivalent to

$$\int_{D_{\lambda,r}} G'^*(z) [W]_{\lambda,r} G'(z) dA(z) \lesssim [W]_{\lambda,r}, \quad \lambda \in \mathbb{D}.$$

Notice that the last relation above can be written as

$$(3.8) \quad \int_{D_{\lambda,r}} \| [W]_{\lambda,r}^{1/2} G'(z) [W]_{\lambda,r}^{-1/2} e \|^2 dA(z) \lesssim \|e\|^2, \quad e \in H, \lambda \in \mathbb{D}.$$

We now claim that (3.8) is equivalent to (3.5). Indeed, using the subharmonicity of $z \mapsto \|[W]_{\lambda,r}^{1/2} G'(z) [W]_{\lambda,r}^{-1/2} e\|$ in (3.8) we get

$$(3.9) \quad (1 - |\lambda|)^2 \|[W]_{\lambda,r}^{1/2} G'(\lambda) [W]_{\lambda,r}^{-1/2} e\|^2 \lesssim \|e\|^2, \quad e \in H, \lambda \in \mathbb{D},$$

and hence one implication is proven. The converse follows by integrating (3.9) with respect to λ on $D_{\zeta,r}$ and using relation (3.7). \square

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